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OF INELASTIC SOLIDS

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E. T. Onat [**]

SUMMARY

The paper is concerned with the representation of mechanical behavior of non-linear solids with memory. The nature and limitations of integral representation of Fréchet are discussed. Definitions of state and state variables, which are based on observable histories, are introduced. These notions give rise to differential equation representation of behavior discussed in the latter part of the paper.

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1. INTRODUCTION

In elastic solids the stress depends on the deformation existing at the time of measurement of the stress. For inelastic solids, such as viscoelastic or plastic materials, this assumption is no longer valid. The current stress in such a solid may depend in some manner and degree upon the deformations which existed in it prior to the time of measurement. From the point of view of classical continuum mechanics the description of mechanical behavior of inelastic solids would therefore involve functional relationships between the stress and the deformation gradient histories. Green and Rivlin [1] and later Green, Rivlin and Spencer [2] have shown that the dependence of the stress on the history of deformation gradient cannot be arbitrary. It is subject to two major restrictions. The first of these arises from the fact that a simultaneous rotation of the deformed body and reference system must leave the stress components unaltered. The second restriction comes from any symmetry which the material may possess in its reference state. These authors have also constructed integral representations of the functional dependence of the stress on the history of deformation gradient. A recent review of modern developments in the continuum mechanics of inelastic solids has been given by Rivlin [3].

The present paper is concerned with a critical review of the subject from the point of view of an experimenter. The above mentioned theoretical work provides the experimenter with the form or structure of the relationship which exists between the histories of stress and strain. The task of filling this structure with physical information is left to the experimenter. It is also up to the experimenter to discover whether a given structure constitutes a convenient tool for the description of mechanical behavior of a given solid. It is hoped that the present survey can be of help to the experimenter in the performance of this double task.

For reasons of simplicity, most of the paper is restricted to the study of a one-dimensional situation. Two brief introductory sections are followed by a

section on the strain response to a simple family of stress histories. This section serves to illustrate a fundamental step in the study of functionals.

It is shown in Sections 4 and 5 that for certain solids Fréchet's integral representation of mechanical behavior would involve a prohibitively large number of multiple integrals for the stress or strain histories of interest. A more convenient representation of mechanical behavior for such solids could be developed by generalizing the notion of piece-wise smooth functions to functionals. An attempt in this direction was recently made by Onat and Wang [4].

Sections 7 and 8 are devoted to a discussion of the differential equation representation of mechanical behavior. This mode of representation, which is intimately connected with the notions of state and state variables, has not as yet received the full attention of the workers in the field, especially in the area of finite deformations.

Section 7 contains a brief discussion of the notions of state and state variables and illustrates the circumstances which allow one to represent a given history dependence by a system of differential equations. It is noted in this section that the differential equations representation constitutes a natural tool for the description of the piece-wise smooth behavior encountered in plastic or nearly plastic solids.

Ordinary plastic solids, such as structural metals at room temperature and at very slow strain rates, are time independent; the current strain in such materials depends on the path of stress history in the stress space, but it is independent of the speed with which the stress point moves on this path. In spite of this simplification it is not convenient to express the path dependence of the strain by means of an integral representation because of lack of analyticity associated with the question of loading and unloading. In structural metals subjected to high strain rates or temperatures the current strain depends on both the stress path and the speed of the stress point on this path and the above

mentioned difficulty associated with loading and unloading continues to exist. The representation of mechanical behavior is more difficult in this case than in ordinary plastic solids; however the differential equation representation for plastic solids which is discussed briefly at the end of the paper can easily be modified to account for the rate effects.

The last section of the paper is concerned with the questions of representation in the case of arbitrary deformations.

2. PHENOMENOLOGICAL STUDY OF MECHANICAL BEHAVIOR

We wish to review briefly the main steps involved in the phenomenological study of mechanical behavior of a solid. For reasons of simplicity we first confine ourselves to the simple case of tensile test which enables one, in principle, to study homogeneous deformations of an initially isotropic solid under the time dependent uniaxial tension. We shall be interested in small isothermal deformations.

The study begins with the preparation of a number of specimens which at the start of testing are identical* in every respect.

A typical experiment involves, if a stress controlling machine is at our disposal, application of time dependent stress $\sigma(\tau)$ on the time interval $[0,T]$ and observation of the resulting time dependent strain $\varepsilon(\tau)$ on the same interval. The result of such an experiment is an input-output pair $(\sigma(\tau)\Big|_0^T, \varepsilon(\tau)\Big|_0^T)$ or for short (σ,ε) on the observation interval $[0,T]$. Different inputs, say, $\sigma^{(1)}$ ... $\sigma^{(n)}$ may be applied to different identical specimens to obtain corresponding outputs $\varepsilon^{(1)}$ ... $\varepsilon^{(n)}$. The result of such a multiple experiment is a set of input-output pairs $(\sigma^{(1)},\varepsilon^{(1)})$ ... $(\sigma^{(n)},\varepsilon^{(n)})$.

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*Reproducibility of tests is a necessary condition for the specimens to be identical. Sufficient conditions for identity are more difficult, if not impossible, to elucidate.

The experimenter would consider that a sufficient insight is gained into the mechanical behavior of the solid by a multiple experiment, if for the stress histories (loading programs) $\sigma(\tau)$ of interest he can guess with reasonable accuracy what the corresponding strain responses $\varepsilon(\tau)$ would be without performing further tests. When this stage is reached the experimenter would like to translate the knowledge on mechanical behavior thus gained into the mathematical language for the purposes of transmitting it to the designer or the analyst.

The task of the experimenter is then to discover and to represent the relationship that exists, for a given inelastic solid, between stress and strain histories.*

Problems similar to the one discussed above arise in the study of communication and control systems under the name of Characterization and Representation Problems or simply the Identification Problem [5]. It is hoped that the workers in mechanics would find it profitable to follow the considerable amount of progress which is being made in the study of the identification problem.

Before closing this section we must point out that the study of the mechanical behavior is often much more difficult than the previous remarks would indicate. For instance in tests involving short times, such as wave propagation and impact experiments, the measurement of stresses is nearly impossible, so that the experiment provides one with wave shapes and speeds measured at various locations instead of a set of input-output pairs (σ, ε). The task of determining mechanical behavior of the solid from the latter evidence is an involved one but it is not unrelated to the class of problems discussed in this paper.

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* A more thorough study of mechanical behavior would involve histories of temperature, heat flow, etc. Inclusion of these would complicate the picture but would introduce no new conceptual difficulties.

3. MATHEMATICAL PRELIMINARIES

We now introduce several basic concepts which are indispensable to the study of input-output pairs. We do this not in the abstract but by referring to the physical situation discussed above.

We start by assuming that at time $\tau = 0$ there is available a set of identical specimens composed of the given solid. The specimens carry no stress at this time and the strains for $\tau \geq 0$ are measured with respect to the configuration at $\tau = 0$ so that

$$\sigma(\tau) = 0, \quad \varepsilon(\tau) = 0 \quad \text{when } \tau = 0. \tag{1}$$

We restrict the attention to the loading programs $\sigma(\tau)$ which are continuous on the observation interval $[0,T]$ and vanish, as (1) indicates, at $\tau = 0$. We shall assume that the strain responses $\varepsilon(\tau)$ corresponding to continuous strain histories $\sigma(\tau)$ are also continuous.[*]

In each test the solid assigns to a given $\sigma(\tau)$ a response $\varepsilon(\tau)$ on the interval $[0,T]$. An electrical engineer would, therefore, consider the solid or the specimen as a black box. A mathematician would, on the other hand, regard the solid as an operator and would use the following notation,

$$\varepsilon = F[\sigma] \tag{2}$$

to express the relationship between continuous input-output pairs defined on $[0,T]$.

One would remain outside the realm of mathematics or physics if one did not assign some properties of regularity to the operator F. Here we shall assume that the operator F for the solid of interest, or for short, the solid F, is continuous. For the purposes of the present paper it may suffice to adopt the following definition of continuity:

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[*]We further assume that the solid is in the zero state at $\tau = 0$; this means that $\sigma(\tau) = 0$ for $\tau \geq 0$ implies $\varepsilon(\tau) = 0$ for $\tau \geq 0$.

It is said that the operator F is <u>continuous</u>, if for each e > 0 there exists a δ > 0 such that the inequality

$$\left| \varepsilon^{(1)}(\tau) - \varepsilon^{(2)}(\tau) \right| < e, \quad \text{on } [0,T] \tag{3}$$

holds for the strain responses produced by any two stress histories which satisfy the condition

$$\left| \sigma^{(1)}(\tau) - \sigma^{(2)}(\tau) \right| < \delta, \quad \text{on } [0,T]. \tag{4}$$

In more physical terms continuity of F implies that the strain responses obtained in tests conducted with slightly different loading programs differ from each other only slightly.*

It may be safe to assert that most solids of interest exhibit this property. However, as is well known, the Voigt solid constitutes an important exception. In this solid slightly different <u>strain histories</u> may produce different stress histories.

Another important property of F follows from the observation that the happenings in the future cannot affect the present and therefore the strain at time t depends only on the stresses on the interval [0,t]. This observation implies that the operator F must have the property of <u>causality</u> expressed by the symbolism

$$\varepsilon(t) = F_{0}^{t}[\sigma(\tau)]. \tag{5}$$

The above equation represents, for a given t, a functional of the stress histories defined on [0,t].

Another property which we may assign to F is related to the question of "aging". The state of the solid at τ = 0 may be such that, under the absence of

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* Here the word "different" is used in the sense of inequalities in (3) and (4).

stress and strain for $\tau \geq 0$, mechanical behavior of the solid may not be affected by the passage of time: that is to say, if $\varepsilon(\tau)$ and $\sigma(\tau)$ is an input-output pair, then the following histories would also constitute an input-output pair:

$$\varepsilon_1(\tau) = \begin{cases} 0 \\ \varepsilon(\tau-x) \end{cases}, \quad \sigma_1(\tau) = \begin{cases} 0 & 0 \leq \tau \leq x \\ \sigma(\tau-x) & x \leq \tau \end{cases} \tag{6}$$

for any non-negative x. Solids which have the above property are called time-invariant or non-aging solids. It can be shown that for time-invariant solids the operator F may be replaced once and for all by a single functional defined on the interval $[0,T]$.

4. STRAIN RESPONSE TO SIMPLE LOADING PROGRAMS

In this section we wish to illustrate several concepts introduced in the previous sections. The present section will also prepare the ground for a discussion of integral representations of mechanical behavior.

We consider the set of loading programs (Fig. 1) defined by

$$\bar{\sigma}(\tau) = \begin{cases} \sigma_1 \dfrac{\tau}{t_1}, & 0 \leq \tau \leq t_1 \\ \sigma_1 + (\sigma_2-\sigma_1) \dfrac{\tau-t_1}{t_2-t_1}, & t_1 \leq \tau \leq t_2 \end{cases} \tag{7}$$

where t_1 and t_2 are given fixed times and σ_1 and σ_2 are arbitrary constants satisfying the inequalities

$$0 \leq \sigma_1 \text{ and } \sigma_2 \leq M. \tag{8}$$

We wish now to focus our attention on the dependence of the strain ε at time t_2 on the stress histories of the type (7). Since each loading program belonging to the above set is fully characterized by the magnitudes σ_1 and σ_2, the continuous functional F in (5) reduces, for argument functions defined in (7), to a

continuous function of σ_1 and σ_2:

$$\varepsilon(t) = \overset{t}{\underset{0}{F}}[\bar{\sigma}(\tau)] = f(\sigma_1, \sigma_2). \tag{9}$$

The above passage from continuous functionals to continuous functions constitutes a key step in the study of functionals.

We must emphasize again that in (7) t_1 and t_2 are kept constant but σ_1 and σ_2 are varied so that the dependence of f on t_1 and t_2 does not enter into the present considerations.

A multiple experiment, composed of n inputs of the type (7) and the corresponding outputs, determines the value of f at n points in (8). If these points are chosen suitably then the experimenter would obtain a very good idea as to what sort of function f is; he would then be able to construct an interpolating function f^* which would constitute an approximate representation of f even over the points which are not covered by the multiple experiment. We note that the choice of n-points may depend on the nature of f (for instance, more experimental points would be needed over the subregions of (8) where f is "steep"). The mathematical representation of the interpolating functions f^* would also depend on the nature of f, as we shall presently see.

We wish now to review the form of $f(\sigma_1, \sigma_2)$ for some basic solids. For linear elastic solids we have, of course,

$$\varepsilon(t) = f(\sigma_1, \sigma_2) = \frac{\sigma_2}{E}. \tag{10}$$

On the other hand, for linear viscoelastic solids

$$f = a_1\sigma_1 + a_2\sigma_2, \tag{11}$$

where a_1 and a_2 are constants. We note that in this case f, the present value of strain, depends on the stress applied at previous instances through the presence

of σ_1 in (11). For most polymers (11) constitutes an excellent approximation for f

provided that σ_1 and σ_2, and hence M, is sufficiently small. For higher values of M,

f becomes non-linear. In some cases the non-linearity in f can be adequately repre-

sented by adding higher order times to (11). For instance an experimental study of

polypropylene fibers by Ward and Onat [6] indicated that up to the extensions of 2%

$$f \stackrel{\sim}{=} a_1\sigma_1 + a_2\sigma_2 + \sum_{i,j,k}^{2} a_{ijk}\sigma_i\sigma_j\sigma_k. \qquad (12)$$

The adequacy of the above representation may indicate that the partial derivatives

of f up to the fourth order are continuous in the domain of interest.

We next consider the rigid-plastic material defined by the stress-strain

diagram shown in Fig. 2 where Y is the yield stress and E_t is the tangent modulus

which for reasons of simplicity is assumed to be constant. The function $f(\sigma_1,\sigma_2)$,

which describes the mechanical behavior of this material for stress histories (7)

is therefore given by

$$f = \begin{cases} 0 & , \quad 0 \le \sigma_1, \quad \sigma \le Y \\ \frac{1}{E_t}(\sigma_1-Y), & \sigma_1 \ge Y, \quad \sigma_1 > \sigma_2 \\ \frac{1}{E_t}(\sigma_2-Y). & \sigma_2 \ge Y, \quad \sigma_2 > \sigma_1 \end{cases} \qquad (13)$$

Note that although f is continuous, its first derivatives exhibit discontinuities

on the interfaces of the domains indicated in (13).

The following remarks concerning f defined in (13) will be useful in the

discussion of integral representations. Since f is continuous it can be approxi-

mated to any degree of accuracy by a polynomial $P(\sigma_1,\sigma_2)$ according to the theorem

of Weierstrass (see [7], p. 481). However the degree of the Weierstrass polynomial

which achieves an acceptable approximation over a closed region of interest may be

very high due to the presence of discontinuities in $\partial f/\partial\sigma_1$ and $\partial f/\partial\sigma_2$ in (13).

To elaborate on this remark we may consider the simple case of Y = 0 and $E_t = 1$

where f becomes:

$$f = \max(\sigma_1, \sigma_2) \tag{14}$$

We now consider the polynomial approximation characterized by the inequality and the domain given below[*]:

$$\left| \max(\sigma_1, \sigma_2) - P(\sigma_1, \sigma_2) \right| < e \quad \text{on } 0 \leq \sigma_1, \sigma_2 \leq 1 \tag{15}$$

It can be shown, using a result of Bernstein [8] concerning the related function $y = |x|$ that the degree N of the polynomial $P(\sigma_1, \sigma_2)$ in (15) must at least be $N \gtrsim \frac{1}{2e}$. This result indicates once more that the Weierstrass polynomial is not a natural tool for the representation of the piece-wise linear function defined in (13).

5. INTEGRAL REPRESENTATION OF MECHANICAL BEHAVIOR

We begin with linear solids. A solid is said to be linear if the operator F representing the mechanical behavior of the solid has the properties of homogeneity and additivity:

(a) $F[\lambda\sigma] = \lambda F[\sigma]$ where λ is a constant,

(b) $F[\sigma^{(1)} + \sigma^{(2)}] = F[\sigma^{(1)}] + F[\sigma^{(2)}]$.

The last property implies that the strain response of the material to the sum of two stress histories is equal to the sum of the strain responses to the individual stress histories (the principle of superposition).

Riesz's well-known work on linear functionals [9] when applied to the present case provides, for differentiable stress histories, the following integral representation:

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[*] It will be assumed throughout the paper that all relevant quantities have been properly non-dimensionalized.

$$\varepsilon(t) = \int_0^t J(t,\tau) \frac{d\sigma}{d\tau} d\tau, \tag{16}$$

where the function $J(t,\tau)$ is independent of the argument function and characterizes the linear solid.

When the solid is time-invariant then J becomes a function of the single argument $(t-\tau)$. The types of experiments needed for the establishment of linearity of a given solid and for the measurement of the associated creep function $J(t)$ or the relaxation modulus $G(t)$ are well-known[*] and will not be discussed here.

Many solids of interest, such as metals subjected to high stresses, temperatures or stress rates, exhibit inelastic behavior but do not possess the properties (a) and (b) of linearity.

Representation of mechanical behavior of such nonlinear solids is a more difficult problem. We have, of course, the following result based on Fréchet's work on nonlinear functionals [11] which is at our disposal for the purposes of representation:

A continuous and causal operator F acting on the set of equicontinuous[**] functions $\sigma(\tau)$ on $[0,T]$ can be represented to any desired degree of accuracy by a sum of multiple integrals:

$$F[\sigma(\tau)] \cong \int_0^t K_1(t,\tau_1)\sigma(\tau_1)d\tau_1 + \ldots$$
$$+ \int_0^t \ldots \int_0^t K_m(t,\tau_1, \ldots, \tau_m)\sigma(\tau_1)\ldots\sigma(\tau_m)d\tau_1 \ldots d\tau_m \tag{17}$$

where the kernels K_i are determined by the operator independently of the argument function $\sigma(\tau)$ and they are continuous functions of τ_i. In the case where F is time-invariant, K_i become continuous functions of $t-\tau_i$.

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[*] For a review of this subject see [10].

[**] For a definition of equicontinuity or compactness see [12].

If we are to use (17) to represent mechanical behavior of a given solid then we must conduct a multiple experiment aimed at the determination of (a) the number of terms to be retained in (17) for a desired degree of accuracy and (b) the kernels defining these terms.

A multiple experiment based on the loading programs (7) discussed in the previous section would shed light on both items. We observe first that for stress histories (7) the sum (17) would become a polynomial of degree m in σ_1 and σ_2. If the experiments suggest that $f(\sigma_1,\sigma_2)$ can be represented adequately by a polynomial, say, of third degree then the indication would be that the first three integrals should be retained in (17). One could then conduct further tests involving more complicated strain histories, say, histories composed of three or four linear segments, to see whether this initial guess remains a good one. Such tests would also help to determine the structure of the relevant kernels K_i.

Studies which follow this approach have been conducted by Ward and Onat [6], Lifshitz [13] and Onaran and Findley [14] for non-linear polymers.

Indications are that for polymers studied in the above-mentioned work, non-linearity in behavior, which occurs when stresses become relatively large, can be adequately represented by the first few terms of the sum (17).

For metals a different situation appears to exist. We have shown in the previous section that for a rigid plastic material the representation of $f(\sigma_1,\sigma_2)$ by a polynomial would require a prohibitively larger number of terms. This would mean therefore that a large number of terms would be needed in the Fréchet sum for the same material. At first sight this would not cause any concern since other means of representation are known for such solids. However, recent work by Wang and Onat [15] on creep of an aluminum alloy at elevated temperatures, showed that for this material the representation of the strain response $\varepsilon(\sigma_1,\sigma_2)$ to loading programs composed of two steps of magnitudes σ_1 and σ_2 would also involve a polynomial of high degree in σ_1 and σ_2. This is partly due to the fact that the

function $\varepsilon(\sigma_1, \sigma_2)$ behaves differently over the domains characterized by the in-equalities $\sigma_1 > \sigma_2$ and $\sigma_2 > \sigma_1$.

This observation indicates that aluminum retains the piece-wise smooth behavior associated with plasticity in spite of the occurrence of viscous effects at high temperature.

The above remarks show that there exist solids of interest for which the Fréchet description of mechanical behavior would involve a large number of terms for stress histories of interest. Such solids may be said to be <u>strongly non-linear</u>. Since a constitutive law of the type (17) containing a large number of terms is likely to be cumbersome in applications, it may be appropriate to look for more convenient representations of mechanical behavior for strongly nonlinear solids.

For this purpose one could try to generalize the notion of piece-wise smooth functions to operators and functionals. Representation of a piece-wise smooth functional would involve a sum of the type (17) containing a few multiple integrals whenever the argument functions lie in a given sub-domain of the space of argument functions. When one goes to an adjacent sub-domain then a different but equally simple integral representation would take over. An attempt in this direction has been made by Onat and Wang in [4], where they discuss order sensi-tive, piece-wise smooth (or linear) functionals.

Representation of mechanical behavior could, in certain cases, be based on a system of differential equations. As will be seen in sections 6 and 7, this mode of representation is intimately connected with the notions of state and of state variables. We will see in section 8 that representation by differential equations is particularly suitable for strongly nonlinear solids.

6. REPRESENTATION OF MECHANICAL BEHAVIOR BY DIFFERENTIAL EQUATIONS IN LINEAR VISCOELASTICITY

In this introductory section and in the sections which will follow, it will be more convenient to regard strain histories as inputs and stress histories as outputs.

We consider a time-invariant linear solid defined by the constitutive law:

$$\sigma(t) = \int_0^t G(t-\tau)\dot\epsilon(\tau)d\tau. \tag{18}$$

We assume that the relaxation modulus $G(t)$ is composed of a sum of exponentials:

$$G(t) = \sum_{i=1}^{n} G_i e^{-t/T_i}, \tag{19}$$

where G_i and T_i are positive constants.

We introduce the following n quantities

$$q_i = \int_0^t e^{-(t-\tau)/T_i}\dot\epsilon(\tau)d\tau, \tag{20}$$

and observe by taking the time derivative of (20) that

$$\dot q_i(t) + \frac{q_i(t)}{T_i} = \dot\epsilon(t). \tag{21}$$

On the other hand a combination of (18), (19) and (20) provides the equation

$$\sigma(t) = \sum_{i=1}^{n} G_i q_i(t). \tag{22}$$

We now observe that (21) and (22), together with the initial conditions

$$q_i = 0 \quad \text{when} \quad t = 0 \tag{23}$$

which follow from (20) can be regarded as the differential equation representation of the mechanical behavior of the solid (18).

It is important to note from (21) and (22) that when q is known at a given

time t, then $\sigma(t)$ can be determined for $\tau \geq t$ from the knowledge of $\varepsilon^{\cdot}(\tau)$ or

equivalently from the knowledge of $\varepsilon(\tau) - \varepsilon(t)$ for $\tau \geq t$. Thus if the strains

are measured for $\tau \geq t$ with respect to the configuration at time t, the behavior

of the solid on [t,T] will depend only on the strains on [t,T] and on the q_i at

time t. This remark suggests that q_i may be called state variables. The existence

of a finite number of state variables implies that the mechanical behavior of

the solid does not depend on all the details of the strain history. As can be

seen from (20), those "components" of the strain rate histories $\varepsilon^{\cdot}(\tau)$ which are

"orthogonal" to the functions e^{τ/T_i} do not contribute to q_i and therefore do not

affect the future behavior of the material.[*]

7. THE CONCEPT OF STATE. REPRESENTATION BY DIFFERENTIAL EQUATIONS.

Consider a set of specimens which have been subjected to some strain

histories on the interval [0,t]. We apply to these specimens further strains

on [t,T]. We denote the strain on [t,T] based on the configuration at time t

by $\varepsilon^{*}(\tau)$. In the present case of small strains we have, of course,

$$\varepsilon^{*}(\tau) = \varepsilon(\tau) - \varepsilon(t), \quad \tau \geq t \tag{24}$$

where $\varepsilon(t)$ and $\varepsilon(\tau)$ are the strains referred to the zero state.

We shall say that a set of specimens are in the same state at time t if

any pair of identical strain histories $\varepsilon^{*}(\tau)$ on [t,T] produce in any two of these

specimens identical stress histories on [t,T].

The test pair shown in Fig. 3 suggests that the strain histories $\varepsilon^{(1)}$ and

$\varepsilon^{(2)}$ on [0,t] may have produced the same state at time t. In order to make sure

that this is actually the case one must conduct further test pairs using the same

strain histories $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ on [0,t], but different $\varepsilon^{*}(\tau)$ on [t,T].

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[*] See [16] for a further discussion of this topic.

A strain history on [0,t] produces a given state at time t. It will be of interest to know the number of states at time t created by all strain histories on [0,t]. This number will, in general, be infinite and some of our future remarks will deal with comparison of infinities.

For an elastic solid all strain histories on [0,t] which have the same terminal value $\varepsilon(t)$ produce the same state at time t. Thus all possible states of an elastic solid at time t can be represented, in the present uniaxial case, by points on the real axis, each point having the coordinate $\varepsilon(t)$. We may therefore say that all possible states of an elastic solid fill the real axis (or a segment of it, if the magnitude of strains of interest is bounded).

Next we consider a Maxwell solid in its familiar model representation (Fig. 4). It can easily be seen from the definition of the state that two strain histories, which at time t give rise to the same elongation in the spring of the model, produce the same state. Therefore all possible states of the Maxwell model also fill the real axis.

On the other hand, for the linear viscoelastic solid considered in section 6, a state is characterized, as we have seen before, by the n quantities q_i so that the state space, for this material, is n-dimensional. The extreme case occurs when every distinct strain-history on [0,t] produces a different state. In this case the state space coincides with the space of input-functions.

We now restrict the discussion to materials which possess an n-dimensional state space. The coordinates of the state point will be denoted by q_i. During a test the state point will move in the state space starting from the point corresponding to the state at $\tau = 0$.

We wish now to prove that the stress at time t must be a function of $q_i(t)$ and t:

$$\sigma(t) = f(q_i(t), t). \tag{25}$$

The proof follows from the definition of state and from the fact that the operator F defining the mechanical behavior of the solid assigns continuous $\sigma(\tau)$ to continuous $\varepsilon(\tau)$ on $[0,T]$.

Consider two strain histories $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ on $[0,T]$. Assume that at time t these histories give rise to the same state. The following continuous strain history

$$\varepsilon(\tau) = \begin{cases} \varepsilon^{(1)}(\tau) \text{ on } [0,t] \\ \varepsilon^{(1)}(t) + \varepsilon^{*(2)}(\tau) \text{ on } [t,T] \end{cases} \tag{26}$$

will produce on $[t,T]$, by the definition of the state, the same stress history as that produced by $\varepsilon^{(2)}$. But the stresses caused at time t by $\varepsilon^{(1)}$ and ε defined in (26) must be the same by the above mentioned property of the operator F. Therefore the stresses at time t produced by $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ must be the same, which proves the assertion.

It can also be shown that if the solid is time-invariant then σ in (25) will depend only on $q_i(t)$.

We now consider the dependence of the state variables q_i on the history of strain. It follows from the definition of state that

$$q_i = F_i[q_j(t),t;\varepsilon^{*}] \text{ on } [t,T] \tag{27}$$

The above symbol means that $q_i(\tau)$ on $[t,T]$ is related by the operator F_i to the history of strain $\varepsilon^{*}(\tau)$ based on the configuration at time t, the operator itself being a function of $q_i(t)$ and t. We may assume that the operator F_i is continuous and causal.

Now we observe that q_i becomes a given function of τ on $[t,T]$ for a given history $\varepsilon^{*}(\tau)$. It is of interest to study the derivative

$$\frac{d}{d\tau} q_i(\tau)_{\tau=t} = q_i^{\circ}(t) \tag{28}$$

It can be shown that if the operator F_i possesses a Fréchet derivative and if $q_i(\tau)$ for $\varepsilon^*(\tau) = 0$ on $[t,T]$ has a derivative at $\tau = t$, then

$$q_i^\circ(t) = g_i(q_j(t),t) + h_i(q_j(t),t)\varepsilon^\circ(t), \qquad (29)$$

where g_i and h_i are functions of the indicated arguments.

Equations (25) and (29) together with the initial conditions $q_i(0) = q_i^0$ constitute a differential equation representation of the mechanical behavior of the solid.

That ε° appears linearly in (29) is due to the existence of the Fréchet derivative of F_i which may be too strong a requirement to place upon F_i for some solids. It is likely that for solids such as metals where the behavior depends strongly on whether there has been loading or unloading in the recent past (29) will have the more general form

$$q_i^\circ(t) = f_i(q_j,\varepsilon^\circ,t) \qquad (30)$$

and f_i will not necessarily be continuous in its arguments.

All that has been said would remain a rather formal mathematical structure if one cannot give broad rules concerning the types of experiments needed for the determination of the <u>number</u> of state variables, and of the functions governing their growth. It would seem that there exists no systematic study of such questions.* It should be remembered however that one rarely encounters an entirely unknown material. With each class of materials there is available a certain amount of knowledge of phenomenological or microscopic nature which provides starting points in such studies. A brief study of plasticity in the next section will illustrate this point.

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* Rules based on simple histories of the type (7) should not be difficult to construct.

8. ARBITRARY DEFORMATIONS

Green and Rivlin [1] have shown that the functional dependence of stress on the deformation gradient history must be of the form[*]

$$\sigma(t) = D(t) \ F \ [\varepsilon(\tau)] \ D^T(t) \tag{31}$$

where σ is the stress tensor, $D(t)$ is the tensor of deformation gradients at time t, $D^T(t)$ is its transpose and F is a tensor valued functional of the strain history on [0,t]. Here the tensor $\varepsilon(t)$ is an appropriate measure of finite strain. The above form is such that a simultaneous rotation of the deformed body and reference frame leaves the stress component unaltered.

In the case where the time-invariant solid is continuous and the components $\varepsilon_{ij}(\tau)$ of the strain history belong to a set of equicontinuous functions a straightforward generalization of (17) provides the following integral representation for F:

$$
F_{ij}[\varepsilon_0^t(\tau)] \cong \int_0^t K_{ijkl}(t-\tau)\varepsilon_{kl}(\tau)d\tau +
$$
$$
+ \int_0^t \int_0^t K_{ijklmn}(t-\tau_1, t-\tau_2)\varepsilon_{kl}(\tau_1)\varepsilon_{mn}(\tau_2)d\tau_1 d\tau_2 + \ldots \tag{32}
$$

If the solid possesses symmetry in its reference state at $\tau = 0$, then the tensor valued kernels in the above expression must exhibit certain properties of invariance. These have been studied exhaustively in recent years (see, for instance, Rivlin [3]).

In the particular case of isotropic solids the above expression takes the following remarkable form [1]:

---

[*] We assume that at $\tau = 0$ elements of the solid are identical and are identically oriented with respect to a fixed coordinate frame.

$$\underset{\sim}{F} \cong K_o \underset{\sim}{I} + \overset{5}{\underset{\beta=1}{\Sigma}} \int_o^t \circ \circ \circ \int_o^t K_\beta (t-\tau_1, \circ \circ \circ, t-\tau_\beta) \tag{33}$$

$$[\underset{\sim}{\varepsilon}(\tau_1) \circ \circ \circ \underset{\sim}{\varepsilon}(\tau_\beta) + \underset{\sim}{\varepsilon}(\tau_\beta) \circ \circ \circ \underset{\sim}{\varepsilon}(\tau_1)] d\tau_1 \circ \circ \circ d\tau_\beta$$

where $\underset{\sim}{I}$ is the unit tensor and K_β's are polynomials in the invariants I_α given by

$$I_\alpha = \int_o^t \cdots \int_o^t \phi^\alpha (t-\tau_1, \circ \circ \circ, t-\tau_\beta) \ \text{tr} \ \underset{\sim}{\varepsilon}(\tau_1) \cdots \underset{\sim}{\varepsilon}(\tau_\beta) \tag{34}$$

$$d\tau_1 d\tau_2 \circ \circ \circ d\tau_\beta$$

these polynomials being dependent on the arguments shown in (33).

If the strains are small enough so that terms of degree, say, higher than the third may be neglected in comparison with those of the third degree, the expression (33) can be simplified considerably [17]. It can easily be seen that the simplified expression contains twelve unknown functions. Lockett [17] has shown how these functions could, in principle, be determined experimentally by means of stress-relaxation experiments. Experimental studies based on this approach have been reported by Lifshitz [13] and Onaran and Findley [14].

The success of this approach depends on the range of validity of the simplified expression. As remarked before, for certain polymers the range of validity may be adequately large, but for metals it may be so small that a different approach to the problem of representation may be needed.

In the discussion of the differential equation representation of mechanical behavior we restrict ourselves to small deformations. For small deformations (small shape changes, small rotations) (31) reduces to

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{F}[\underset{\sim}{\varepsilon}\overset{t}{\underset{o}{(\tau)}}], \tag{35}$$

where $\underset{\sim}{\varepsilon}$ now is the strain tensor employed in the classical theory of elasticity.

If one considers a time-invariant material equations (25) and (30) take

the form[*]

$$\sigma_{ij}(t) = f_{ij}(q_\alpha(t)),$$

$$q_\alpha^\circ(t) = f_\alpha(q_\beta(t), \varepsilon_{ij}^\circ(t)),$$

(36)

where σ_{ij} and ε_{ij} are, respectively, the components of the stress and strain

tensors, in a fixed rectangular frame.

An important question which must be faced at this stage is concerned with

the possible tensorial character of the state variables. The state variables can

be regarded as average quantities describing the microscopic structure of the

solid and therefore it would be natural to expect that they will be tensor valued.

It can also be argued that it would be more convenient to work with tensor

valued state variables. For instance, if the state variables are tensor valued

then one could apply recent results [18] on the form invariance to (36) to obtain

a representation appropriate to isotropic solids.

We now consider briefly the representation of elastic-plastic behavior.[**]

It is convenient in this case to regard the stress histories as inputs. Constitu-

tive equations for an elastic-plastic solid have the following structure, which

is similar to (36):

$$\varepsilon_{ij}(t) = C_{ijkl}\sigma_{kl}(t) + f_{ij}(q_\alpha(t)),$$

(37)

$$q_\alpha^\cdot = \begin{cases} f_{\alpha kl}(\sigma_{ij}(t), q_\beta(t))\sigma_{kl}^\cdot(t) & \text{(a)} \\ 0 & \text{(b)} \end{cases}$$

(38)

The first term in (37) represents elastic components of the strain, the second

represents the plastic components. The latter depend on the history of stress

---

[*] Generalization of the definitions of state and state variables to the present
case of arbitrary small deformations is self-evident and will not be given here.

[**] For a detailed discussion of the stress-strain relations in Plasticity, see [19].

and hence are functions of the state variables q_α.

In (38) the case (a) occurs when the current stress point is on the yield surface and is moving out of it at the instant of interest:

$$f(\sigma_{ij}(t), q_\alpha(t)) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0. \qquad \text{(a)}$$

The function f which defines the yield surface depends on the history through the state variables q_α.

The case (b) occurs when (i) the stress point is within the yield surface or (ii) the stress point is on the yield surface, but it is not moving out:

$$f < 0, \quad \text{or} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \leq 0. \qquad \text{(b)}$$

It may be of interest to comment here on the number n of state variables. In a similar treatment of Plasticity, Kröner [20] uses, as state variables, the components of a symmetric second order tensor which is a measure of the dislocation loop density. Green and Naghdi's treatment of Plasticity [21] involves, in the present context, seven state variables, six of them being the components of the plastic strain tensor.

In his theory of strain-hardening, Prager [22] assumes that the yield surface moves in the stress space with no change in shape and orientation. Therefore Prager's theory involves six state variables.

Experiments are, of course, the ultimate source of information on the number of state variables. A multiple experiment based on different stress histories will produce a family of yield surfaces. The number of parameters needed for a (approximate) description of this family can be taken, as a first guess, as the number of state variables.

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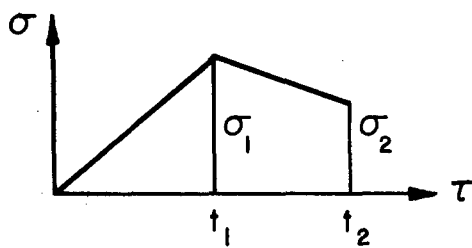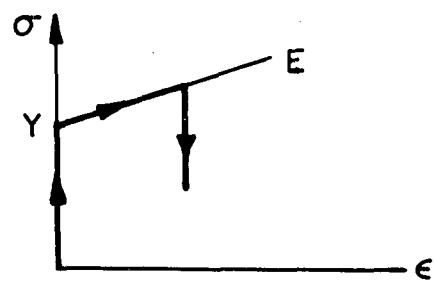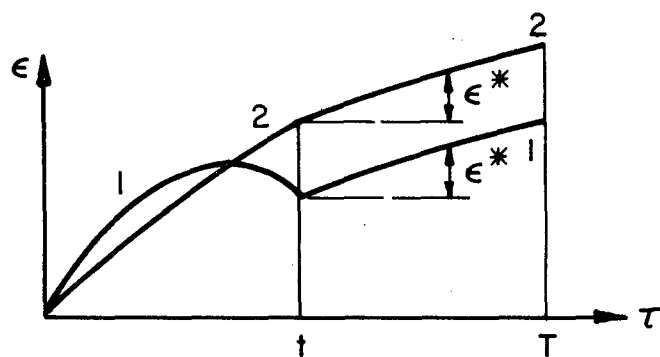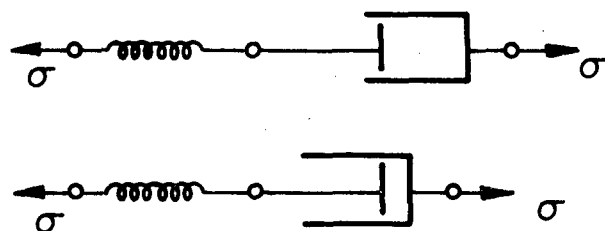
FIG. 1

FIG. 2

FIG. 3

FIG. 4